SOLUTION OF SOME PERIODIC ELASTOPLASTIC PROBLEMS

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The article considers an elastoplastic problem for a plane weakened by an infinite number of round openings. It is assumed that the level of the stresses and the distance between the openings are such that the round openings are completely enveloped by the corresponding plastic zone; under these circumstances, the adjacent plastic regions do not coalesce. The article also considers the inverse elastoplastic problem under conditions of plane strain for an unbounded plane, weakened by a periodic series of openings. A number of communications have been devoted to periodic problems in the theory of elasticity and plasticity with an unknown boundary [1-8]. In distinction from [1-8], in which the method of perturbations was used, another method is used to solve periodic elastoplastic problems, making it possible to obtain a solution with any arbitrary relative dimensions of the region.

§1. Let there be a plane with round openings, having a radius $R(R<1)$ and centers at the points $P_m = m\omega (m = 0, \pm 1, \pm 2, \ldots)$, $\omega = 2$.

We introduce the notation: L_m is the contour of an opening with its center at the point P_m; Γ_m is the corresponding elastoplastic boundary; D_z is the exterior of the contour Γ_m.

At the contour of an opening L_m the boundary conditions have the form

$$\sigma_r = -p, \quad \tau_{r\theta} = 0.$$

The condition of creep is taken in the form

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2.$$

The field of the stresses in the plastic zone has the form [9]

$$\sigma_r = -p + 2k \ln \frac{r}{R}; \quad \sigma_\theta = 2k - p + 2k \ln \frac{r}{R}, \quad \tau_{r\theta} = 0. \tag{1.1}$$

In the elastic region the stresses are determined using the Kolosov–Muskhelishvili formula [10]:

$$\sigma_r + \sigma_\theta = 4\,\mathrm{Re}\,\Phi(z); \tag{1.2}$$
$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2[\bar{z}\Phi'(z) + \Psi(z)]e^{2i\theta}.$$

At the unknown contour Γ_m, separating the elastic and plastic regions, all the stresses are continuous. Using formulas (1.1), (1.2), we obtain the boundary conditions at the contour Γ_m:

$$\mathrm{Re}\,\Phi(z) = \frac{k-p}{2} + \frac{k}{2} \ln \frac{z\bar{z}}{R^2}; \quad \bar{z}\Phi'(z) + \Psi(z) = k\frac{\bar{z}}{z}.$$

We go over to the parametric plane of the complex variable ζ using the transform $z = \omega(\zeta)$. The analytical function $z = \omega(\zeta)$ effects the conformal mapping of the region D_z on the region D_ζ in the plane of ζ, which is the exterior of circles l_m of radius λ with centers at the points P_m.

To determine the three analytical functions $\varphi(\zeta) = \Phi[\omega(\zeta)]$, $\psi(\zeta) = \Psi[\omega(\zeta)]$ and $\omega(\zeta)$, we obtain a nonlinear boundary-value problem at l_m:

$$\mathrm{Re}\,\varphi(\zeta) = \frac{k-p}{2} + \frac{k}{2} \ln \frac{\omega(\zeta)\overline{\omega(\zeta)}}{R^2}; \tag{1.3}$$

$$\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) + \psi(\zeta) = k\frac{\overline{\omega(\zeta)}}{\omega(\zeta)}. \tag{1.4}$$

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Solving the Dirichlet problem (1.3), we find that at D_ζ

$$\varphi(\zeta)=\frac{p-k}{2}+k\ln\frac{\omega(\zeta)}{R}-k\ln\frac{\zeta}{\lambda}. \tag{1.5}$$

Using (1.5), we transform boundary condition (1.4) at l_m to the form

$$\zeta\omega'(\zeta)\Psi(\zeta)=\overline{k\omega(\zeta)}. \tag{1.6}$$

The functions $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$ are sought in the form of the series

$$\varphi(\zeta)=\alpha_0+\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k)}(\zeta)}{(2k+1)!}; \tag{1.7}$$

$$\psi(\zeta)=\sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k)}(\zeta)}{(2k+1)!}-\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}s^{(2k+1)}(\zeta)}{(2k+1)!}; \tag{1.8}$$

$$\omega(\zeta)=\zeta+\sum_{k=0}^{\infty}A_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k-1)}(\zeta)}{(2k+1)!}, \tag{1.9}$$

where
$$\rho(\zeta)=\left(\frac{\pi}{\omega}\right)^2\frac{1}{\sin^2\left(\frac{\pi}{\omega}\zeta\right)}-\frac{1}{3}\left(\frac{\pi}{\omega}\right)^2;\ s(\zeta)=\sum_m{}'\left[\frac{P_m}{(\zeta-P_m)^2}-\frac{2\zeta}{P_m^2}-\frac{1}{P_m}\right].$$

Primes with the summation sign mean that the index m = 0 is included in the summation.

We not give the dependences which must be satisfied by the coefficients of expressions (1.7)-(1.9). From the conditions of symmetry with respect to the coordinate axes we find that

$$\operatorname{Im}\alpha_{2k+2}=\operatorname{Im}\beta_{2k+2}=\operatorname{Im}A_{2k+2}=0,\ k=0,\ 1,\ 2,\ldots \tag{1.10}$$

It can be shown that the relationships (1.7)-(1.10) define a class of symmetrical problems with a periodic distribution of the stresses [11].

From the condition of the equality to zero of the principal vector of the forces acting on the arc connecting two congruent points in D_ζ, it follows that

$$\alpha_0=\frac{\pi^2}{24}\beta_2\lambda^2.$$

By virtue of the satisfaction of the conditions of periodicity, the system of boundary conditions (1.6) at l_m (m = 0, ±1, ±2, ...) is replaced by one functional equation, for example, at the contour l_0.

To set up equations for the remaining coefficients of the functions $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$, we expand these functions in Laurent series in the neighborhood of the point $\zeta = 0$:

$$\varphi(\zeta)=\alpha_0+\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}}{\zeta^{2k+2}}+\sum_{k=0}^{\infty}\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}r_{j,k}\zeta^{2j}; \tag{1.11}$$

$$\psi(\zeta)=\sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}}{\zeta^{2k+2}}+\sum_{k=0}^{\infty}\beta_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}r_{j,k}\zeta^{2j}-\sum_{k=0}^{\infty}(2k+2)\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}(2j+2k+2)r_{j,k}\zeta^{2j}; \tag{1.12}$$

$$\omega(\zeta)=\zeta-\sum_{k=0}^{\infty}A_{2k+2}\frac{\lambda^{2k+2}}{(2k+1)}\frac{1}{\zeta^{2k+1}}+\sum_{k=0}^{\infty}A_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}\frac{r_{j,k}\zeta^{2j+1}}{2j+1}; \tag{1.13}$$

$$r_{j,k}=\frac{(2j+2k+1)!\,g_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}},\quad g_{j+k+1}=2\sum_{m=1}^{\infty}\frac{1}{m^{2j+2k+2}}.$$

Substituting into the boundary conditions (1.3), (1.6) at the contour $l_0(\zeta=\lambda e^{i\theta})$, in place of $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$, their expansions (1.11)-(1.13), and equating the coefficients with $e^{2ik\theta}$ (k = 0, ±1, ±2, ...), we obtain an infinite system of nonlinear algebraic equations with respect to α_{2k}, β_{2k}, A_{2k} [condition (1.3) was first differentiated with respect to θ].

The equations of the first approximation have the form

$$c\beta_2+A_2\beta_4\lambda^4r_{1,0}+A_2\gamma_0=kc;\quad c\beta_4+A_2\beta_2=\frac{1}{3}kA_2\lambda^4r_{1,0};$$
$$c\gamma_0+A_2\gamma_1+A_2\beta_2\lambda^4r_{1,0}=-kA_2;\quad 2\alpha_2(1+\lambda^4r_{1,0})d=d_1,$$

where

$$c = 1 + A_2\lambda^2 r_{0,0};\ d = c^2 + \left(1 + \frac{1}{9}\lambda^{10} r_{1,0}^2\right) A_2^2;$$

$$d_1 = -2cA_2\left(1 - \frac{1}{3}\lambda^4 r_{1,0}\right);\ \gamma_j = \beta_2 r_{j,0}\lambda^{2j+2} + \beta_4 r_{j,1}\lambda^{2j+4} - 2(2j+2)\alpha_2\lambda^{2j+2} r_{j,0}\quad (j = 0, 1).$$

To obtain expressions connecting the parameter λ with the applied load p, we substitute formulas (1.7), (1.9) into the boundary condition (1.3), we multiply the expression obtained by $1/2\pi i\zeta$, and integrate over the circular contour l_0. As a result, we obtain [10]

$$\alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2}\lambda^{2k+2} r_{0,k} = \frac{k-p}{2} + k\ln\frac{\lambda}{R}\left[1 + \sum_{k=0}^{\infty} A_{2k+2}\lambda^{2k+2} r_{0,k}\right].$$

The results of a calculation in the first two approximations are given in Table 1.

Setting $\zeta = \lambda e^{i\theta}$ in (1.13), we obtain the equation of the elastoplastic boundary:

$$r = |\omega(\lambda e^{i\theta})| = f(\theta).$$

In the first approximation

$$r^2 = \lambda^2(d + d_1\cos 2\theta). \tag{1.14}$$

Here

$$r_{\max} = \lambda\left[1 + A_2\left(-1 + \lambda^2\sum_{j=0}^{\infty}\frac{r_{j,0}}{2j+1}\lambda^{2j}\right)\right];$$

$$r_{\min} = \lambda\left[1 + A_2\left(1 + \lambda^2\sum_{j=0}^{\infty}\frac{(-1)^j r_{j,0}}{2j+1}\lambda^{2j}\right)\right]. \tag{1.15}$$

Figure 1 shows the elastoplastic boundary for the case R = 0.3, p = 2.12 (λ = 0.7, r_{max} = 0.81, r_{min} = 0.58).

The condition $r_{min} \geq R$ determines the smallest load with which the contour of an opening is completely enveloped by the plastic zone. The relationship (1.15), with $r_{max} \leq 1$, permits finding the greatest load with which the plastic zones touch each other.

Figure 2 gives the dependences of the parameter λ on the value of the applied load p/k for several values of the radius of an opening R. Up to now, the mean stresses in the plane were assumed equal to zero. In the plane, let there be the mean stresses (elongational at infinity)

$$\sigma_x + \sigma_x^\infty,\ \sigma_y = \sigma_y^\infty,\ \tau_{xy} = 0.$$

In this case the solution is sought in the form

$$\varphi_*(\zeta) = \frac{\sigma_x^\infty + \sigma_y^\infty}{4} + \varphi(\zeta);$$

$$\psi_*(\zeta) = \frac{\sigma_y^\infty - \sigma_x^\infty}{2} + \psi(\zeta),$$

where $\varphi(\zeta)$ and $\psi(\zeta)$ are defined by relationships (1.7), (1.8).

TABLE 1

Coefficients	λ = 0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8
				First approximation				
α_2/k	0,00796	0,02902	0,05677	0,08503	0,11005	0,13038	0,14587	0,15693
β_2/k	1,00006	1,00085	1,00330	1,00774	1,01409	1,02269	1,03508	1,05521
β_4/k	0,00796	0,02909	0,05733	0,08726	0,11607	0,14352	0,17129	0,20281
A_2	−0,00796	−0,02905	−0,05697	−0,08592	−0,11279	−0,13709	−0,16003	−0,18362
				Second approximation				
α_2/k	0,00796	0,02914	0,05766	0,08820	0,11759	0,14498	0,17228	0,18701
α_4/k	0,00021	0,00239	0,00889	0,01882	0,02468	0,02504	0,02597	0,02672
β_2/k	1,00006	1,00084	1,00331	1,00778	1,01420	1,02306	1,03621	1,05787
β_4/k	0,00796	0,02902	0,05677	0,08582	0,11176	0,13612	0,16801	0,18033
β_6/k	−0,00014	−0,00154	−0,00555	−0,01036	−0,01073	−0,00293	0,00814	0,01169
A_2	−0,00796	−0,02905	−0,05697	−0,08592	−0,11290	−0,13778	−0,16182	−0,18594
A_4	0,00001	0,00009	0,00028	0,00001	−0,00208	−0,00702	−0,01321	−0,01674

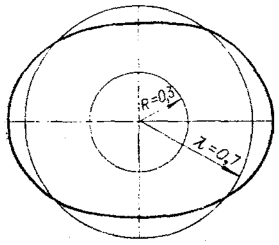

Fig. 1

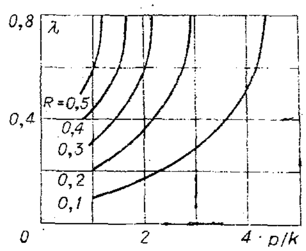

Fig. 2

TABLE 2

Coefficients	λ							
	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8
				First approximation				
β_2/p	1,00822	1,03286	1,07362	1,12932	1,19717	1,27224	1,34808	1,41607
β_4/p	—0,00828	—0,03395	—0,07921	—0,14709	—0,23994	—0,35651	—0,48724	—0,60592
A_2	0,00822	0,03293	0,07445	0,13401	0,21518	0,32659	0,48933	0,76102
				Second approximation				
β_2/p	1,00822	1,03286	1,07360	1,12918	1,19636	1,26899	1,33927	1,40162
β_4/p	—0,00829	—0,03398	—0,07949	—0,14864	—0,24588	—0,37395	—0,53021	—0,70450
β_6/p	0,00002	0,00042	0,00224	0,00760	0,02034	0,04695	0,09695	0,18274
A_2	0,00822	0,03293	0,07445	0,13401	0,21509	0,32600	0,48672	0,75238
A_4	0,00004	0,00067	0,00342	0,01088	0,02704	0,05838	0,11845	0,24735

§2. Let there be a plane, weakened by unknown curvilinear openings having centers at the points $P_m = m\omega$ ($m = 0, \pm 1, \pm 2, \ldots$), $\omega = 2$.

We denote the contour of an opening with its center at the point P_m by L_m, and the exteriors of the contours L_m by D_z. At the unknown contour of an opening L_m the boundary conditions have the form

$$\sigma_n = -p; \quad \tau_{nt} = 0; \quad \sigma_t = \sigma_* = \text{const} \tag{2.1}$$

(t and n are the directions of the tangent and the normal to the contour of the body).

In the case of an elastic body, the value of $\sigma_* = \text{const}$ is subject to determination during the process of solution of the problem. For an elastoplastic material, the relationship $\sigma_t = \sigma_*$ is a condition imposed on the development of the plastic zone, i.e., it reduces to the requirement that, at the moment of origin, the plastic zone embrace the whole contour of an opening at the same time, without penetrating into the depths of the body. In this case, $\sigma_* = \text{const}$ is a given value, for example, under the conditions of plane strain $\sigma_t = \sigma_* = -p \pm 2k$.

We go over to the parametric plane ζ using the transform $z = \omega(\zeta)$. The analytical function $z = \omega(\zeta)$ effects the conformal mapping of the region D_z on D_ζ in the plane ζ, which is the exterior of circles Γ_m of radius λ with centers at the points P_m.

On the basis of the equalities [10]

$$\sigma_n + \sigma_t = \sigma_x + \sigma_y \quad (\zeta = \lambda e^{i\theta});$$

$$\sigma_t - \sigma_n + 2i\tau_{nt} = \frac{\zeta^2}{\lambda^2} \frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}} (\sigma_y - \sigma_x + 2i\tau_{xy})$$

and the boundary conditions (2.1), for determining the three analytical functions $\varphi(\zeta)$, $\psi(\zeta)$, and $\omega(\zeta)$, we obtain a nonlinear boundary-value problem at Γ_m:

$$\operatorname{Re} \varphi(\zeta) = a; \tag{2.2}$$

$$\zeta^2 \left[\overline{\omega(\zeta)}\, \varphi'(\zeta) + \omega'(\zeta)\, \psi(\zeta)\right] = \lambda^2 b \overline{\omega'(\zeta)} \tag{2.3}$$

$$\left(a = \frac{\sigma_* - p}{4}, \quad b = \frac{\sigma_* + p}{2}\right).$$

The boundary condition (2.3) can be transformed. Solving the Dirichlet problem (2.2) we find that, in D_ζ,

$$\varphi(\zeta) = a. \tag{2.4}$$

Using (2.4), we write the boundary condition (2.3) at Γ_m in the form

$$\zeta^2 \omega'(\zeta)\, \psi(\zeta) = \lambda^2 b\, \overline{\omega'(\zeta)}. \tag{2.5}$$

We seek the functions $\psi(\zeta)$ and $\omega(\zeta)$ in the form of the series (1.8), (1.9); in the series (1.8) the coefficients α_{2k} ($k=1, 2, \ldots$) are identically equal to zero.

From the condition of the equality to zero of the principal vector of the forces acting on an arc connecting two congruent points in D_ζ, it follows that

$$a = \frac{\pi^2}{24}\beta_2\lambda^2.$$

To set up equations for the remaining coefficients of the series (1.8), (1.9) of the functions $\psi(\zeta)$ and $\omega(\zeta)$, we expand these functions in Laurent series in the neighborhood of the point $\zeta=0$. Substituting into the boundary condition (2.5) at the contour $\Gamma_0(\zeta=\lambda e^{i\theta})$, in place of $\psi(\zeta)$, $\omega(\zeta)$, and $\overline{\omega'(\zeta)}$, their expansions in Laurent series, and equating coefficients with $e^{2ik\theta}$ ($k=0, \pm1, \pm2, \ldots$), we obtain an infinite system of nonlinear algebraic equations for β_{2k}, A_{2k}.

The results of a calculation in the first two approximations are given in Table 2.

In the first approximation the equation of the equal-strength form of an opening has the form (1.14).

The constant σ_* is expressed in the form

$$\sigma_* = \frac{\pi^2}{6}\beta_2\lambda^2 + p. \tag{2.6}$$

For an elastoplastic body, the relationship (2.6) is the condition for the solvability of the problem.

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